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Fractional relaxation in anelastic solids

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Abstract

The ordinary relaxation phenomenon exhibiting a pure exponential decay is generalized by replacing the first time derivative by the α-fractional derivative ($0<\alpha\leqslant 1$) in the basic equation. Mathematical aspects are discussed with emphasis on the related continuous relaxation spectrum. From the physical point of view the thermoelastic coupling in anelastic solids is considered to take into account a temperature fractional relaxation due to diffusion. A viscoelastic model, formerly introduced by Caputo and Mainardi, is then recovered which generalizes the standard linear solid.

1. Introduction

The relaxation process is very common and is found in many areas of physics; in particular, it plays a fundamental role in the physical interpretation of anelasticity in solids. From the mathematical point of view this process is governed by a simple linear differential equation of the first order in time.

The purpose of this paper is to consider, after some mathematics, the physical aspects of relaxation in anelastic solids when in the basic equation the first-order derivative is replaced by a derivative of fractional order α with $0<\alpha<1$; the generalized process will be referred to as *fractional relaxation*.

The fractional derivative is an integrodifferential operator which provides an analytical extension of the ordinary concept of the derivative of integer order. Its mathematical treatment is found in the so-called *fractional calculus* which is the generalization of the ordinary calculus to take into account integration and differentiation of order not necessarily integer [1].

Here we limit ourselves to consider *causal* functions, *i.e.* functions of a real variable t vanishing for $t<0$. For these functions the fractional derivative of order α with $0<\alpha\leqslant 1$ results as

$$\frac{\mathrm{d}^\alpha}{\mathrm{d}t^\alpha} f(t) = \frac{1}{\Gamma(1-\alpha)} \frac{\mathrm{d}}{\mathrm{d}t} \int_0^t \frac{f(\tau)}{(t-\tau)^\alpha}\, \mathrm{d}\tau$$

$$= \frac{1}{\Gamma(1-\alpha)} \int_0^t \left(\frac{\mathrm{d}f}{\mathrm{d}\tau}\right) \frac{\mathrm{d}\tau}{(t-\tau)^\alpha} + f(0^+)\Phi_{(1-\alpha)}(t) \qquad (1)$$

with

$$\Phi_{(1-\alpha)}(t) = \frac{t^{-\alpha}}{\Gamma(1-\alpha)}\Theta(t) = \frac{t_+^{-\alpha}}{\Gamma(1-\alpha)} \qquad (2)$$

where Γ denotes the gamma function and Θ the Heaviside step function. For details we refer to our recent report [2].

2. The mathematical aspects

By the relaxation equation we mean the first-order differential equation

$$\frac{\mathrm{d}u}{\mathrm{d}t} + q_0 u(t) = 0 \qquad \text{with } u(0)=u_0>0 \qquad (3)$$

where $q_0>0$ and the field variable $u=u(t)$ is assumed to be differentiable (and therefore continuous) in all of $\mathbb{R}$. The solution of (3)

$$u(t) = u_0 \exp(-q_0 t) \qquad (4)$$

can be easily obtained using standard methods; the constant q_0 is referred to as the relaxation frequency, being the inverse of the relaxation time.

From the physical point of view the field variable is expected to be a *causal* function of t, *i.e.* vanishing for $t<0$ with a possible jump at $t=0$, so that it may be convenient to introduce the notation

$$f(t) = u(t)\Theta(t) \qquad \text{with } f(0^+) = u(0) = u_0 \qquad (5)$$

As a consequence we rewrite (3) in the form

$$\frac{\mathrm{d}f}{\mathrm{d}t} + q_0 f(t) = u_0 \delta(t) \qquad (6)$$

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where $\delta(t)$ denotes the Dirac delta (generalized) function. This equation is to be interpreted in the generalized sense of the distribution theory. Its solution, obtained by the generalized Laplace transform technique, reads

$$f(t)=u_0 \exp(-q_0 t)\Theta(t) \tag{7}$$

By replacing the first derivative in (3) and (6) by a derivative of order α with $0<\alpha<1$ and taking into account (1), we are led to consider the following equations for $u(t)$ (in the ordinary sense) and $f(t)$ (in the generalized sense):

$$\frac{1}{\Gamma(1-\alpha)}\int_0^t \left(\frac{du}{d\tau}\right)\frac{d\tau}{(t-\tau)^\alpha}+q^\alpha u(t)=0 \tag{8}$$

with $u(0)=u_0$, and

$$\frac{d^\alpha f}{dt^\alpha}+q^\alpha f(t)=u_0\Phi_{(1-\alpha)}(t) \tag{9}$$

For dimensional convenience the parameter q_0 has been replaced by q^α, where q is the generalized relaxation frequency. We note that for $\alpha=1$ equations (3) and (6) are recovered provided that $q=q_0$, recalling that $\Phi_0(t)=t_+^{-1}/\Gamma(0)=\delta(t)$.

The solution can be obtained by using the (generalized) Laplace transform technique on (8) or (9). The image solution turns out to be, in an obvious notation,

$$\bar{u}(s)=\tilde{f}(s)=u_0\frac{s^{\alpha-1}}{s^\alpha+q^\alpha} \tag{10}$$

To invert (10), we can use (see *e.g.* refs. 2 and 3) the expansion theorem or the Bromwich formula; we obtain

$$u(t)=u_0 E_\alpha[-(qt)^\alpha] \tag{11}$$

where E_α denotes the Mittag–Leffler function defined by [4]

$$E_\alpha(z)=\sum_{k=0}^{\infty}\frac{z^k}{\Gamma(\alpha k+1)}=\frac{1}{2\pi i}\int_{Ha}\frac{\zeta^{\alpha-1}e^\zeta}{\zeta^\alpha-z}\,d\zeta \tag{12}$$

with $\alpha\geqslant 0$, $z\in\mathbb{C}$ and Ha denoting the Hankel loop. From the series representation we recognize that for $\alpha>0$, $E_\alpha(z)$ is an *entire function*, which provides a simple generalization of the exponential function. Furthermore, for $0\leqslant\alpha\leqslant 1$, $E_\alpha(-x)$ turns out to be *completely monotonic* for $x\geqslant 0$, *i.e.*

$$(-1)^n\frac{d^n}{dx^n}E_\alpha(-x)\geqslant 0 \tag{13}$$

This property thus makes $E_\alpha(-x)$ a good candidate to generalize the pure exponential decay into a more general relaxation process introducing the additional parameter α in a simple way. In this respect earlier applications were given by Davis [5] and Gross [6]. Later Caputo and Mainardi [3, 7] and more recently several authors, including Koeller [8, 9], have shown the relevance of this function when derivatives of real order α $(0<\alpha<1)$ are introduced in the stress–strain relation for linear viscoelastic solids. In the next section we shall provide for these models an original interpretation based on the thermoelastic coupling, thus extending a classical argument due to Zener [10].

Plots of the fractional relaxation function $E_\alpha[-(qt)^\alpha]$ *vs.* qt for different values of α appear interesting, overall in comparison with the ordinary relaxation function $\exp(-qt)$. At first we can derive the asymptotic behaviour for $t\to 0^+$ and $t\to\infty$ respectively from the series and integral representation in (12); they read

$$E_\alpha[-(qt)^\alpha]\approx\begin{cases}1-\dfrac{(qt)^\alpha}{\Gamma(1+\alpha)} & \text{as } qt\to 0^+\\[2ex] \dfrac{(qt)^{-\alpha}}{\Gamma(1-\alpha)} & \text{as } qt\to\infty\end{cases} \tag{14}$$

An explicit expression for the fractional relaxation function can be obtained for the intermediate value $\alpha=0.5$, which reads

$$E_{0.5}[-(qt)^{1/2}]=\exp(qt)\,\mathrm{erfc}[(qt)^{1/2}] \tag{15}$$

A plot of the function (15) *vs.* qt is presented in Fig. 1 along with its asymptotic expressions (14), in comparison with the ordinary relaxation function $\exp(-qt)$. With respect to the exponential function the fractional relaxation function exhibits for small t a much faster decay (its derivative tends to $-\infty$ in comparison with -1), but for large t a much slower decay (algebraic decay in comparison with exponential decay).

To conclude the mathematical analysis, we would like to point out that, because of the property (13), the fractional relaxation function can be expressed in

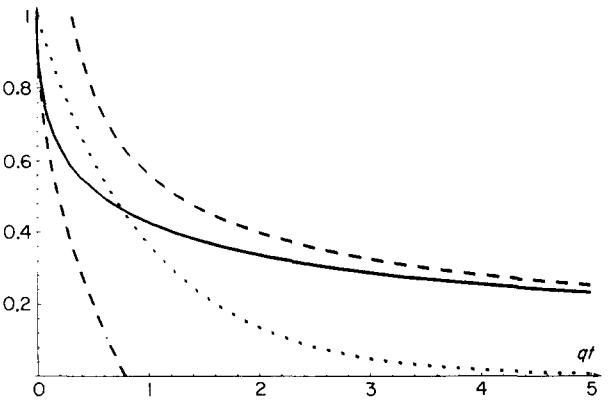

Fig. 1. Comparison between the fractional relaxation function $E_{0.5}[-(qt)^{1/2}]$ given by (15) (continuous curve) and the ordinary relaxation function $\exp(-qt)$ (dotted curve) *vs.* qt. The asymptotic expressions of (15) are also reported (dashed curve).

terms of a *relaxation distribution* function or *relaxation spectrum* $\xi((\tau;\ \alpha,\ q)$, *i.e.*

$$E_\alpha[-(qt)^\alpha]=\int_0^\infty \exp\left(-\frac{t}{\tau}\right)\xi(\tau;\ \alpha,\ q)\ \mathrm{d}\tau \tag{16}$$

In order to obtain this spectrum, we can adopt the general method of Gross [6] involving Stieltjes transform (see *e.g.* ref. 3) or we can directly use the integral representation in (12) [2]. We get

$$\xi(\tau;\ \alpha,\ q)=\frac{1}{\pi\tau}\frac{\sin(\pi\alpha)}{(q\tau)^\alpha+(q\tau)^{-\alpha}+2\cos(\pi\alpha)} \tag{17}$$

A plot of this spectrum, scaled with q, *vs.* $q\tau$ is shown in Fig. 2 for several values of α. We recognize that the spectrum is a decreasing function of $q\tau$ for $0<\alpha<\alpha_*$, where $\alpha_*\approx 0.736$ is the solution of the transcendental equation $\alpha=\sin(\alpha\pi)$; then, with increasing α, it exhibits a minimum and a maximum before tending to the impulsive function $\delta(q\tau-1)$ as $\alpha\to 1$.

3. The physical aspects

According to Zener [10], the physical interpretation of anelasticity in metals is linked to a spectrum of relaxation phenomena. In particular, the thermal relaxation due to diffusion in the thermoelastic coupling is essential to derive the standard constitutive equation (stress–strain relation) in linear viscoelasticity. This equation corresponds to a simple rheological model (with three independent parameters) known as the standard linear solid (SLS); it reads

$$\sigma+\tau_\epsilon\frac{\mathrm{d}\sigma}{\mathrm{d}t}=M_\mathrm{r}\left(\epsilon+\tau_\sigma\frac{\mathrm{d}\epsilon}{\mathrm{d}t}\right) \tag{18}$$

where $\sigma=\sigma(t)$ and $\epsilon=\epsilon(t)$ denote the uniaxial stress and strain respectively. The three parameters are M_r,

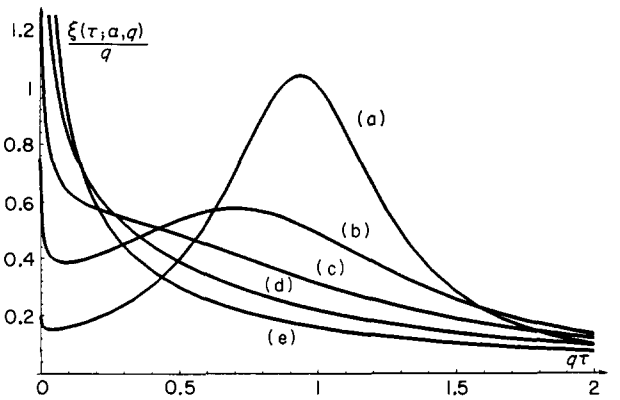

Fig. 2. Plot of the relaxation spectrum (17) of the fractional relaxation function (11), scaled with q, *vs.* $q\tau$ for various values of α: (a) 0.9; (b) 0.8; (c) 0.7; (d) 0.6; (e) 0.5.

which represents the relaxed modulus, and τ_σ and τ_ϵ, which denote the relaxation times under constant stress and strain respectively; an additional parameter is the unrelaxed modulus M_u given by $\tau_\sigma/\tau_\epsilon=M_\mathrm{u}M_\mathrm{r}>1$.

Following Zener, the model equation (18) can be derived from the basic equations of the thermoelastic coupling provided that τ_σ and τ_ϵ also represent the relaxation times for temperature relaxation at constant stress and strain respectively and M_r and M_u represent the isothermal and adiabatic moduli respectively. Denoting by ΔT the deviation of the temperature from its standard value, the two basic equations of thermoelasticity are

$$\epsilon=\frac{1}{M_\mathrm{r}}\sigma+\lambda\Delta T \tag{19}$$

$$\frac{\mathrm{d}}{\mathrm{d}t}\Delta T=-\frac{1}{\tau_\epsilon}\Delta T-\gamma\frac{\mathrm{d}\epsilon}{\mathrm{d}t} \tag{20}$$

where λ is the linear thermal expansion coefficient and $\gamma=(\partial T/\partial\epsilon)_{\mathrm{adiab}}$. Equation (20) results from the combination of the two basic phenomena which induce temperature changes, (a) relaxation due to diffusion

$$\left(\frac{\mathrm{d}}{\mathrm{d}t}\Delta T\right)_{\mathrm{diff}}=-\frac{1}{\tau_\epsilon}\Delta T \tag{21}$$

and (b) adiabatic strain change

$$\left(\frac{\mathrm{d}}{\mathrm{d}t}\Delta T\right)_{\mathrm{adiab}}=-\gamma\frac{\mathrm{d}\epsilon}{\mathrm{d}t} \tag{22}$$

Putting $1+\lambda\gamma=\tau_\sigma/\tau_\epsilon=M_\mathrm{u}/M_\mathrm{r}$ and eliminating ΔT between (19) and (20), the relation (18) is readily obtained. In this way the temperature plays the role of a hidden variable.

If now we assume that the relaxation due to diffusion is governed by the fractional differential equation

$$\left(\frac{\mathrm{d}^\alpha}{\mathrm{d}t^\alpha}\Delta T\right)_{\mathrm{diff}}=-\frac{1}{\tilde{\tau}_\epsilon^{\,\alpha}}\Delta T,\qquad 0<\alpha\leqslant 1 \tag{23}$$

where $\tilde{\tau}_\epsilon$ is a suitable relaxation time, we allow for a natural generalization of the simple process of relaxation, which now depends on the parameter α. As a consequence, equation (20) turns out to be modified into

$$\frac{\mathrm{d}^\alpha}{\mathrm{d}t^\alpha}\Delta T=-\frac{1}{\tilde{\tau}_\epsilon^{\,\alpha}}\Delta T-\gamma\frac{\mathrm{d}^\alpha\epsilon}{\mathrm{d}t^\alpha} \tag{24}$$

and, *mutatis mutandis*, the stress–strain relation turns out to be

$$\sigma+\tilde{\tau}_\epsilon^{\,\alpha}\frac{\mathrm{d}^\alpha\sigma}{\mathrm{d}t^\alpha}=M_\mathrm{r}\left(\epsilon+\tilde{\tau}_\sigma^{\,\alpha}\frac{\mathrm{d}^\alpha\epsilon}{\mathrm{d}t^\alpha}\right) \tag{25}$$

where we have used $1+\lambda\gamma=(\tilde{\tau}_\sigma/\tilde{\tau}_\epsilon)^\alpha=M_\mathrm{u}/M_\mathrm{r}$. By varying

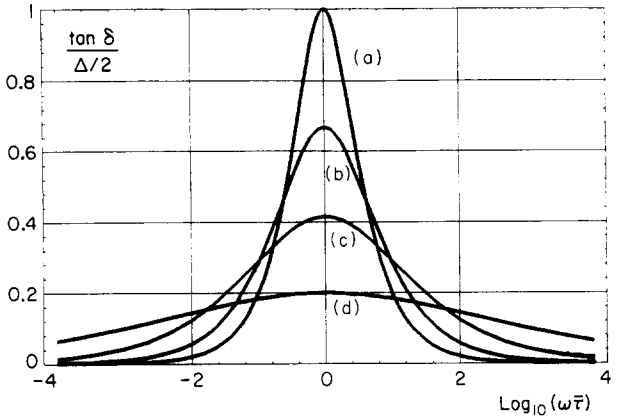

Fig. 3. Plot of the loss tangent (27)–(29) (with $\beta=1$), scaled with $\Delta/2$, *vs.* $\log_{10}(\omega\bar{\tau})$ for various values of α: (a) 1; (b) 0.75; (c) 0.5; (d) 0.25.

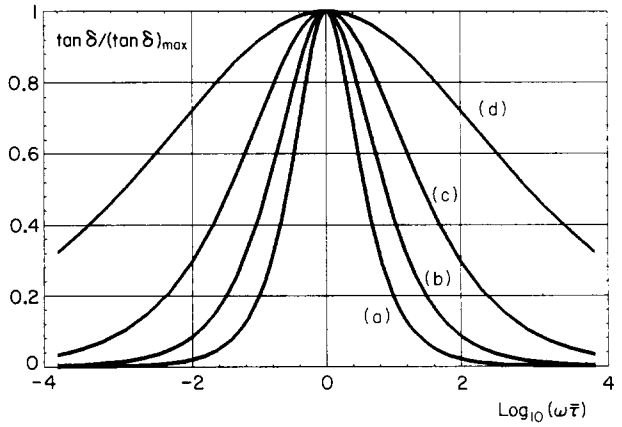

Fig. 4. Plot of the loss tangent (27)–(29) (with $\beta=1$), scaled with its maximum, *vs.* $\log_{10}(\omega\bar{\tau})$ for various values of α: (a) 1; (b) 0.75; (c) 0.5; (d) 0.25.

α, the relation (25) provides a class of rheological models which exhibit, at constant strain $\epsilon(t)=\epsilon_0\Theta(t)$, the stress relaxation law

$$\sigma(t;\alpha)=M_r\epsilon_0+(M_u-M_r)\epsilon_0 E_\alpha\left[-\left(\frac{t}{\bar{\tau}_\epsilon}\right)^\alpha\right] \tag{26}$$

Equation (25) was first introduced by Caputo and Mainardi [7] as an empirical model to fit data on internal friction *vs.* frequency.

As a measure of internal friction we take the *loss tangent, i.e.* the tangent of the angle δ by which strain lags behind stress under sinusoidal excitation of radian frequency ω. It results as

$$\tan\delta=\Delta\,\frac{(\omega\bar{\tau})^\alpha\sin(\alpha\pi/2)}{1+(\omega\bar{\tau})^{2\alpha}+2\beta(\omega\bar{\tau})^\alpha\cos(\alpha\pi/2)} \tag{27}$$

where

$$\Delta=\frac{M_u-M_r}{(M_uM_r)^{1/2}}=\frac{\bar{\tau}_\sigma^{\,\alpha}-\bar{\tau}_\epsilon^{\,\alpha}}{\bar{\tau}^\alpha} \tag{28}$$

$$\bar{\tau}=(\bar{\tau}_\sigma\bar{\tau}_\epsilon)^{1/2},\qquad \beta=\frac{\bar{\tau}_\sigma^{\,\alpha}+\bar{\tau}_\epsilon^{\,\alpha}}{2\bar{\tau}^\alpha} \tag{29}$$

When the loss tangent is plotted against the logarithm of $\omega\bar{\tau}$, it is seen to be a symmetrical function around the maximum attained at $\omega\bar{\tau}=1$. It reduces to the classical Debye peak when $\alpha=1$. Because of the low experimental values of Δ, it is permissible to put $\beta=1$ in (27)–(29), so that the maximum of the loss tangent turns out to be

$$(\tan\delta)_{\max}=\frac{\Delta}{2}\,\frac{\sin(\alpha\pi/2)}{1+\cos(\alpha\pi/2)} \tag{30}$$

For fixed Δ the peak decreases in amplitude and broadens at a rate depending on α, as shown in Fig. 3. For the sake of convenience, in view of applications to experimental data, in Fig. 4 we report the normalized loss tangent obtained when the maximum amplitude is kept constant.

4. Conclusions

We have shown how the fractional calculus can be used in generalizing the physical law of thermal relaxation in anelastic solids to yield the so-called fractional relaxation of order α with $0<\alpha\leqslant 1$. The resulting effect is to introduce a class of intermediate rheological models depending on α, which range from the limiting elastic model ($\alpha=0$) to the classical SLS ($\alpha=1$). The mechanical relaxation properties are readily expressed by a Mittag–Leffler function in terms of a continuous spectrum of relaxation times. The internal friction *vs.* driving frequency turns out to be broader than the Debye peak exhibited by the SLS, with a rate depending on α.

We hope that these models can be successfully adopted to interpret experimental data on relaxation and internal friction in materials of novel physical interest.

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